Special Relativity: Intuition and More

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Invariant Spacetime Interval

SR is about events and the Spacetime Interval.

Examples of "events" include:

- Explosions at some place and time
- Existence of something at some place and time
- A measurement of a field value at some space and time

Fundamental Axiom of Special Relativity: The S.I. between 2 events A and B remains invariant across all inertial observers.

$$S.I. \equiv -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \quad (1)$$

$$\text{where } \Delta t \equiv t_A - t_B \text{ and similar for } x, y, z \quad (2)$$

Transformation

Mathematically, what are the set of coordinate transformations that leave $-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ invariant?

- Time Reversal

$$\mathcal{T} : t \mapsto -t \tag{3}$$

- Parity

$$\mathcal{P}_x : x \mapsto -x \tag{4}$$

likewise y, z

- Time Translations

$$H : t \mapsto t + a \tag{5}$$

- Space Translations

$$p_x : x \mapsto x + a \tag{6}$$

likewise y, z

Transformation

Mathematically, what are the set of coordinate transformations that leave $-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ invariant?

- **Spatial** Rotations $SO(3)$

$$\mathbf{R}_X(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \quad \mathbf{R}_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_Z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R}(\phi,\theta,\psi) = \mathbf{R}_X(\phi)\mathbf{R}_Y(\theta)\mathbf{R}_Z(\psi)$$

  Spatial rotations leave $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ invariant and t_A, t_B untouched.

- Hyperbolic **Spacetime** Rotations (Lorentz Boosts) $SO(3,1)$
  Trigonometry becomes Hyperbolic Trigo (sinh, cosh, tanh)

Physical Interpretation

From mathematical constrains, 2 frames observing the events **can** possibly

- TR: be going in opposite directions in time
- Parity: see the world mirrored
- Time Translations: start their clocks later
- Space Translations: be at different locations
- Spatial Rotations: be oriented differently
- Lorentz Boosts: move at **different velocities**

Let's investigate Lorentz Boosts in greater detail.

Rotations

Let's start off with rotations/Lorentz boosts in 2D. We can extend it to higher dimensions by composing these rotations later

$$\mathbf{R}(\phi, \theta, \psi) = \mathbf{R}_X(\phi)\mathbf{R}_Y(\theta)\mathbf{R}_Z(\psi)$$

The 2D rotation matrix is

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{7}$$

One can check that

$$R(\theta)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

has the same length as before

$$(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 \tag{8}$$

$$= x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta \tag{9}$$

$$+ x^2\sin^2\theta + 2xy\cos\theta\sin\theta + y^2\cos^2\theta \tag{10}$$

$$= x^2 + y^2 \quad (\text{using } \cos^2\theta + \sin^2\theta = 1) \tag{11}$$

Hyperbolic Rotations

If length was defined as $x^2 - y^2$ instead of $x^2 + y^2$, what would rotations look like? Answer: Hyperbolic Trigo!

$$\Lambda(w) = \begin{bmatrix} \cosh w & -\sinh w \\ -\sinh w & \cosh w \end{bmatrix} \tag{12}$$

One can check that

$$\Lambda(w) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cosh w - y\sinh w \\ -x\sinh w + y\cosh w \end{bmatrix}$$

has the same (redefined, hyperbolic) length as before

$$(x\cosh w - y\sinh w)^2 - (-x\sinh w + y\cosh w)^2 \tag{13}$$
$$= x^2\cosh^2 w - 2xy\cosh w\sinh w + y^2\sinh^2 w \tag{14}$$
$$-\ (x^2\sinh^2 w - 2xy\cosh w\sinh w + y^2\cosh^2 w) \tag{15}$$
$$= x^2 - y^2 \quad (\text{using } \cosh^2 w - \sinh^2 w = 1) \tag{16}$$

Geometrical Picture

Rotation vs Hyperbolic Rotation

https://www.desmos.com/calculator/jujhsy4q1t

Physical Interpretation

Considering the hyperbolic rotations on **spacetime** i.e. (ct, x) instead of (x, y), it corresponds to changing between frames of different velocity. Why? Let's see desmos

Connection with Velocity

Considering hyperbolic rotation on $(ct, x) = (1, 0)$,

$$\Lambda(w) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cosh w \\ -\sinh w \end{bmatrix} \tag{17}$$

This point $(ct, x) = (\cosh w, -\sinh w)$ has inverse slope $x/ct = -\tanh w$. Interpreting inverse slope as velocity $\beta \equiv v/c$, we interpret hyperbolic rotations as boosting from a stationary frame to some moving frame. Velocity β is therefore related to the "hyperbolic angle" w by

$$\beta = \tanh w \tag{18}$$

. w is also known as the rapidity.

Velocity Addition

Boosting with rapidity w, followed by another boost with rapidity u? What is the resultant matrix?

$$\Lambda(w)\Lambda(u) = \begin{bmatrix} \cosh w & -\sinh w \\ -\sinh w & \cosh w \end{bmatrix} \begin{bmatrix} \cosh u & -\sinh u \\ -\sinh u & \cosh u \end{bmatrix} \quad (19)$$

$\mid$ exercise: use hyperbolic trigo identities e.g. $\cosh(a+b)$

$$= \begin{bmatrix} \cosh(w+u) & -\sinh(w+u) \\ -\sinh(w+u) & \cosh(w+u) \end{bmatrix} \quad (20)$$

$$= \Lambda(w+u) \quad (21)$$

So boosting twice with velocities $\tanh w$ and $\tanh u$ gives a frame with velocity of $\tanh(w+u)$. If we express $\tanh(w+u)$ in terms of the other velocities (using tanh addition formula), we get **relativistic velocity addition**,

$$\tanh(w+u) = \frac{\tanh w + \tanh u}{1 + \tanh w \tanh u} \quad (22)$$

$$v_1 \oplus v_2 = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2} \quad (23)$$

Lorentz Factor

Recovering the Lorentz factor γ:

$$\gamma \equiv \cosh w \tag{24}$$

$$= \frac{1}{\sqrt{1 - \tanh w}} \tag{25}$$

$$= \frac{1}{\sqrt{1 - \beta^2}} \tag{26}$$

So the Lorentz boost looks like

$$\begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \tag{27}$$

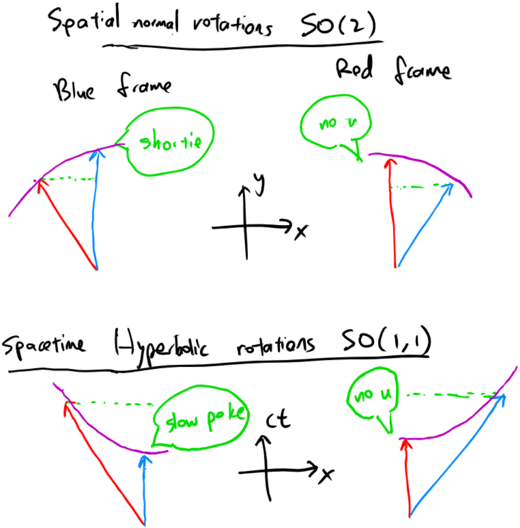
Lorentz Transformation

In the full 4 dimensions, a lorentz boost along x coordinate is

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{28}$$

Time Dilation

Best shown with desmos



Length Contraction

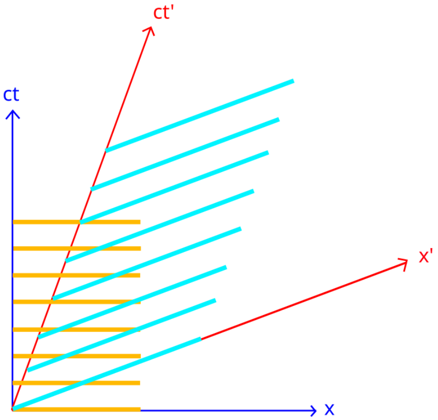


Figure: Orange stick sitting still becomes cyan moving stick

Wait? Isn't that cyan stick longer? Why is it length contraction and not length **dilation**

Length Contraction

Answer is that lengths are measured at equal time!

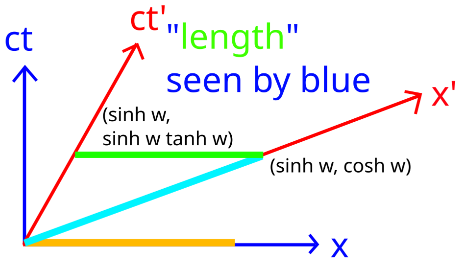


Figure: Length is measured at equal time slices, which is dependent on frame!

Exact length is $\cosh w - \sinh w \tanh w = \sqrt{1-\beta^2}$, which reproduces the factor we learn in length contraction.

Loss of Simultaneity

If both ends of the stick explodes at the same time in one frame A, it will not be at the same time in another moving frame B of velocity v. Moreover, in another frame C of velocity $-v$, the order of events would be swapped when compared to B. Complete loss of simultaneity!

Light Cone

Is causality lost though? Fortunately not.

Question: If two events A, B are causally linked (e.g. A causes B), will boosting to another frame cause B to occur before A?

Answer: No, draw a light cone for A. All Lorentz transformations will keep B within A's light cone. Causality is saved.

In fact, causality is defined using the spacetime interval $-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$,

$$S.I. > 0 \quad \text{(spacelike)} \tag{29}$$

$$S.I. = 0 \quad \text{(lightlike)} \tag{30}$$

$$S.I. < 0 \quad \text{(timelike)} \tag{31}$$

There exist a frame s.t. spacelike events are at the same location.
There exist a frame s.t. timelike events are at the same time.
In all frames, lightlike events lie on the light cone.

Proper Time

A (general) curve through spacetime M_4 can be parameterized. It's basically a map

$$\lambda : \mathbb{R} \to M_4 \tag{32}$$

We can choose any parameterization we want in general.

The most natural parameterization (why?) for a constant velocity curve is the proper time, which is the rate at which the time ticks for the observer in his own frame.

$$\lambda_{\text{inertial}} : \mathbb{R} \to M_4 \tag{33}$$

$$\lambda_{\text{inertial}} : \tau \mapsto X^\mu(\tau) := (ct(\tau), \vec{\mathbf{x}}(\tau)) \tag{34}$$

$$t(\tau) = \frac{\tau}{\sqrt{1-\beta^2}} \tag{35}$$

$$\vec{\mathbf{x}}(\tau) = \vec{\mathbf{v}}t(\tau) = \frac{\vec{\mathbf{v}}\tau}{\sqrt{1-\beta^2}} \tag{36}$$

Kinematics (Four-Vectors)

The proper time is invariant under Lorentz transformations.

Consequence: The proper time τ is the most natural because then derivatives wrt τ transform under Lorentz transformations!

$$\frac{dX^{\mu}}{d\tau} \mapsto \Lambda(w)\frac{dX^{\mu}}{d\tau} \tag{37}$$

So with τ parameterization of the curve, our four-vectors (such as $P^{\mu} \equiv mV^{\mu}$) transform linearly.

If we had chosen an alternative parameterization (such as dX/dt), then the transformations would be nonlinear, and that's sad because we can't use Linear Algebra anymore.

Bless 4-Vectors

Very powerful machinery for SR. For example, we can derive general velocity addition easily using 4-vectors

$$U = \begin{pmatrix} \gamma_u c \\ u\gamma_u \cos\alpha \\ u\gamma_u \sin\alpha \\ 0 \end{pmatrix} \tag{38}$$

$$U' = \Lambda U = \gamma_u \begin{pmatrix} \left(1 - \left(uv/c^2\right)\cos\alpha\right)\gamma_v c \\ (u\cos\alpha - v)\gamma_v \\ u\sin\alpha \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \gamma_{u'} c \\ u'\gamma_{u'}\cos\alpha' \\ u'\gamma_{u'}\sin\alpha' \\ 0 \end{pmatrix}$$

Also, in relativistic dynamics, we always add 4-momentum together. Underlying all this freedom is Linear Algebra.

$$\Lambda(p+q) = \Lambda p + \Lambda q \tag{39}$$

Dynamics

We want to define an action that is Lorentz invariant. At the moment we only have a point particle (spin 0) which transforms trivially (doesn't transform) under Lorentz, so the only sensible action is

$$S = -mc^2 \int d\tau \tag{40}$$

where $-(cd\tau)^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2$. m is the rest mass, and the reason for $-$ is quite nuanced. Then, the integral can be reparameterized using some chosen frame

$$\begin{aligned} S &= -mc^2 \int dt \frac{d\tau}{dt} && (41) \\ &= -mc^2 \int dt \sqrt{1 - \frac{\vec{v}^2}{c^2}} && (42) \end{aligned}$$

Motivating Momentum

$$S = -mc^2 \int dt \sqrt{1 - \frac{\vec{v}^2}{c^2}} \tag{43}$$

$$\frac{d}{dt} \underbrace{\frac{\partial L}{\partial v_i}}_{=:p_i} = \frac{\partial L}{\partial x_i} \tag{44}$$

$$\frac{d}{dt}\left(\frac{mv_i}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}\right) = 0 \tag{45}$$

Since $\vec{p} = \gamma m \vec{v} = m\frac{d\vec{x}}{d\tau}$, if we want to define a 4-momentum to be consistent with this 3-momentum, we have to define $p^0 = m\frac{d(ct)}{d\tau} = \gamma mc$. So all together, 4-momentum

$$P^\mu \equiv mV^\mu \tag{46}$$

$$\text{where } V^\mu \equiv \frac{d}{d\tau}(ct, x, y, z) \tag{47}$$

Motivating Energy

The first component of 4-momentum is γmc, but how do we know this is E/c? Back to Lagrangian mechanics!

$$H \equiv \sum_{i=1}^{3} v_i p_i - L \tag{48}$$

$$= \frac{m\vec{v}^2}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} + mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}} \tag{49}$$

$$= \frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} m\left(\vec{v}^2 + c^2\left(1-\frac{\vec{v}^2}{c^2}\right)\right) \tag{50}$$

$$= \gamma mc^2 \tag{51}$$

Hence we make the connection that energy is just 0th component of 4-momentum!

$$P^\mu = \left(P^0, P^1, P^2, P^3\right) = \left(\frac{E}{c}, p_x, p_y, p_z\right) = m\frac{d}{d\tau}(ct, x, y, z) \tag{52}$$

Derivatives

The calculation is abit involved (deeper understanding would require studying differential geometry), but turns out derivatives transform under the inverse of Lorentz transformation. The math is basically multivariate chain rule.

$$\Lambda : x \mapsto w \tag{53}$$

$$w = \Lambda x \tag{54}$$

$$\Lambda^{-1} w = x \tag{55}$$

$$\left(\Lambda^{-1}\right)^{\rho}{}_{\mu} w^{\mu} = x^{\rho} \tag{56}$$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} \mapsto \frac{\partial}{\partial w^{\mu}} = \frac{\partial x^{\rho}}{\partial w^{\mu}} \frac{\partial}{\partial x^{\rho}} \tag{57}$$

$$= \left(\Lambda^{-1}\right)^{\rho}{}_{\mu} \frac{\partial}{\partial x^{\rho}} \tag{58}$$

$$= \left(\Lambda^{-1}\right)^{\rho}{}_{\mu} \partial_{\rho} \tag{59}$$

Motivating Electromagnetism

How do we get 4-current J^μ and 4-potential A^μ? It comes from studying the "properties" of the Lorentz group. Specifically,

Questions: What fields can we define that we can make a (more interesting) Lorentz invariant action out of (instead of just $\int d\tau$)?

Answer: We can define fields that 4-vectors, aka they transform under Lorentz transformations as

$$A^\mu \mapsto A'^\mu \tag{60}$$

$$A'^\mu(x) \equiv \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) \tag{61}$$

This is in contrast with the spin-0 fields

$$\phi \mapsto \phi' \tag{62}$$

$$\phi'(x) \equiv \phi(\Lambda^{-1}x) \tag{63}$$

Analogy with Rotation Group

Draw a picture

Making Lorentz Invariant Quantities

If you have two 4-vectors V^μ, W^μ, you can form a Lorentz invariant "dot product"

$$V^\mu W_\mu \equiv \left(V^0, V^1, V^2, V^3\right) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} W^0 \\ W^1 \\ W^2 \\ W^3 \end{pmatrix} \tag{64}$$

This is analogous to how angles are invariant under rotation.

Likewise, the inner product of two 4-vectors is invariant under hyperbolic spacetime rotations.

And... Action!

Particle of charge q in electromagnetic field A^μ

$$S = -mc^2 \int d\tau + q \int A_\mu dx^\mu \tag{65}$$

$$\Rightarrow \quad \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{66}$$

Maxwell Equations

$$S = \int d^4x - \frac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right) + \frac{1}{2}\left(\partial_\mu A^\mu\right)^2 - A_\mu J^\mu \tag{67}$$

Topological theta term ($E \cdot B$ term) in Axion Electrodynamics

$$S = \theta \int d^4x \partial_\mu \left(\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma\right) \tag{68}$$

Your creativity is the limit! But wait there's more

Spinors

The symmetry group that the universe has is not actually the Lorentz group $SO(3,1)$, but a double cover of that $SL(2,\mathbb{C})$. This means that we still have our 4-vectors that we developed so far. But we have additional stuff too!

These new spinors we can define are 2-component complex vectors (not our usual notion of 4-component four-vectors).

$$\psi^\alpha = \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} = \begin{pmatrix} a + bi \\ c + di \end{pmatrix} \tag{69}$$

Boost and Rotation of 4-Vectors

We previously saw that spacetime/4-vectors gets rotated by Lorentz transformations and spatial rotations as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{70}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & \\ 0 & \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{71}$$

The above is a boost in the x direction with rapidity w, and a rotation about the y axis with angle θ.

Boost and Rotation of Spinor

The transformation for a spinor under Lorentz boosts and rotations are

$$\begin{bmatrix} \psi^{0\prime} \\ \psi^{1\prime} \end{bmatrix} = \begin{bmatrix} \cosh(w/2) & -\sinh(w/2) \\ -\sinh(w/2) & \cosh(w/2) \end{bmatrix} \begin{bmatrix} \psi^{0} \\ \psi^{1} \end{bmatrix} \tag{72}$$

$$\begin{bmatrix} \psi^{0\prime} \\ \psi^{1\prime} \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \begin{bmatrix} \psi^{0} \\ \psi^{1} \end{bmatrix} \tag{73}$$

The above is a boost in the x direction with rapidity w, and a rotation about the y axis with angle θ.

We see what people mean by "a spinor must rotate 720° to get back" now!

Theoretical Physics

https://en.wikipedia.org/wiki/Representation_theory_of_the_Lorentz_group

Irreducible representations for small (m, n).

Dimension in parenthesis.

	$m = 0$	$\frac{1}{2}$	1	$\frac{3}{2}$
$n = 0$	Scalar (1)	Left-handed Weyl spinor (2)	Self-dual 2-form (3)	(4)
$\frac{1}{2}$	Right-handed Weyl spinor (2)	4-vector (4)	(6)	(8)
1	Anti-self-dual 2-form (3)	(6)	Traceless symmetric tensor (9)	(12)
$\frac{3}{2}$	(4)	(8)	(12)	(16)

General Relativity

Idea of locality + Show black hole metric