Alternative Derivation

$$\text{Res}[\epsilon(z)T(z)\phi(w)] = \delta\phi(w) = h(\partial_w\epsilon(w))\phi(w) + \epsilon(w)\partial_w\phi(w)$$

$$\text{LHS} = \frac{1}{2\pi i}\oint \epsilon(z)T(z)\phi(w)dz \tag{57}$$

$$\text{RHS} = \frac{h\phi(w)}{2\pi i}\oint \frac{\epsilon(z)}{(z-w)^2}dz + \frac{\epsilon(w)}{2\pi i}\oint \frac{\phi(z)}{(z-w)^2}dz \tag{58}$$

$$= \frac{1}{2\pi i}\oint \frac{h\phi(w)\epsilon(z)}{(z-w)^2} + \frac{\epsilon(w)\phi(z)}{(z-w)^2}dz \tag{59}$$

We might be tempted to conclude that

$$\epsilon(z)T(z)\phi(w) = \frac{h\phi(w)\epsilon(z)}{(z-w)^2} + \frac{\epsilon(w)\phi(z)}{(z-w)^2} \tag{60}$$

But it's only true under the contour integral (which selects a specific singular term). So the above expression is only true for some terms (we will show that "some" is $(z-w)^{-1}$ and $(z-w)^{-2}$)

Alternative Derivation

$$T(z)\phi(w) := \ldots + \frac{a_{-3}}{(z-w)^3} + \frac{a_{-2}}{(z-w)^2} + \frac{a_{-1}}{z-w} + \ldots \tag{61}$$

We have $\epsilon(z) \in \text{span}(\epsilon, \epsilon z, \epsilon z^2)$,

Setting $\epsilon(z) \propto 1$, we get $T\phi$ to (-1)st order

$$\text{LHS} = \frac{1}{2\pi i}\oint T(z)\phi(w)dz = a_{-1} \tag{62}$$

$$\text{RHS} = \frac{1}{2\pi i}\oint \underbrace{\frac{h\phi(w)}{(z-w)^2}}_{0} + \underbrace{\frac{\phi(z)}{(z-w)^2}}_{\partial_w \phi(w)} dz \tag{63}$$

$$= \partial_w \phi(w) \tag{64}$$

Alternative Derivation

$$T(z)\phi(w) := \ldots + \frac{a_{-3}}{(z-w)^3} + \frac{a_{-2}}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w} + \ldots \tag{61}$$

We have $\epsilon(z) \in \mathrm{span}(\epsilon, \epsilon z, \epsilon z^2)$,

Setting $\epsilon(z) \propto z$, we get $T\phi$ to (-2)nd order

$$\text{LHS} = \frac{1}{2\pi i} \oint zT(z)\phi(w)dz \tag{62}$$

$$= \frac{1}{2\pi i} \oint \underbrace{(z-w)T(z)\phi(w)}_{a_{-2}} + w \underbrace{T(z)\phi(w)}_{\partial_w \phi(w)} dz \tag{63}$$

$$= a_{-2} + w\partial_w \phi(w) \tag{64}$$

Alternative Derivation

$$T(z)\phi(w) := \ldots + \frac{a_{-3}}{(z-w)^3} + \frac{a_{-2}}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w} + \ldots \tag{61}$$

We have $\epsilon(z) \in \text{span}(\epsilon, \epsilon z, \epsilon z^2)$,

Setting $\epsilon(z) \propto z$, we get $T\phi$ to (-2)nd order

$$\text{RHS} = \frac{1}{2\pi i} \oint \frac{h\phi(w)z}{(z-w)^2} + \frac{w\phi(z)}{(z-w)^2} dz \tag{62}$$

$$= \frac{1}{2\pi i} \oint \underbrace{\frac{h\phi(w)(z-w)}{(z-w)^2}}_{h\phi} + \underbrace{\frac{h\phi(w)w}{(z-w)^2}}_{0} + \underbrace{\frac{w\phi(z)}{(z-w)^2}}_{w\partial_w\phi(w)} dz \tag{63}$$

$$= h\phi(w) + w\partial_w\phi(w) \tag{64}$$

Alternative Derivation

$$T(z)\phi(w) := \dots + \frac{a_{-3}}{(z-w)^3} + \frac{a_{-2}}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w} + \dots \quad (61)$$

We have $\epsilon(z) \in \text{span}(\epsilon, \epsilon z, \epsilon z^2)$,

Setting $\epsilon(z) \propto z$, we get $T\phi$ to (-2)nd order

$$\text{LHS} = a_{-2} + w\partial_w \phi(w) \quad (62)$$

$$\text{RHS} = h\phi(w) + w\partial_w \phi(w) \quad (63)$$

Comparing LHS = RHS,

$$a_{-2} = h\phi(w) \quad (64)$$

Alternative Derivation

$$T(z)\phi(w) := ... + \frac{a_{-3}}{(z-w)^3} + \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w} + ... \quad (61)$$

We have $\epsilon(z) \in \text{span}(\epsilon, \epsilon z, \epsilon z^2)$,

Setting $\epsilon(z) \propto z^2$, we get $T\phi$ to (-3)rd order

$$z^2 = (z-w)^2 + 2w(z-w) + w^2 \quad (62)$$

$$\text{LHS} = a_{-3} + 2wa_{-2} + w^2 a_{-1} \quad (63)$$

$$= a_{-3} + 2wh\phi(w) + w^2 \partial_w \phi(w) \quad (64)$$

$$\text{RHS} = 2wh\phi(w) + w^2 \partial_w \phi(w) \quad (65)$$

Comparing LHS = RHS,

$$a_{-3} = 0 \quad (66)$$

Alternative Derivation

$$T(z)\phi(w) = \ldots + \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w} + \ldots \tag{67}$$