Propagators: Massive Scalar Field Theory

More details at tch1001.github.io

Starting with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + J\phi \xrightarrow{\text{Euler-Lagrange}} \left(\partial^\mu \partial_\mu + m^2\right)\phi = J$$

Analogous to $\mathbf{A}\vec{x} = \vec{b}$

Solved using Green's Functions

"Invert" $(\partial^2 + m^2)$

$$\left(\partial^2 + m^2\right) G(x, x') = \delta^{(4)}(x - x') \longrightarrow \phi(x) = \int d^4x' \, G(x, x') \, J(x')$$

Analogous to $\vec{x} = \mathbf{A}^{-1}\vec{b} = \sum_j (\mathbf{A}^{-1})_{ij} \vec{b}_j$

Fourier Transform to deal with the derivatives

In k-space, $G(x, x') = \int d^4k \; e^{-ik^\mu (x_\mu - x'_\mu)} g(k)$

$$\left(\partial^2 + m^2\right) \int d^4k \; e^{-ik(x-x')} g(k) = \int d^4k \; e^{-ik(x-x')}$$

$$\left(-k^\mu k_\mu + m^2\right) g(k) = 1$$

$$g(k) = \frac{1}{-\omega^2 + \vec{k}^2 + m^2}$$

$$g(k) = \frac{-1}{(\omega - E_k)(\omega + E_k)}$$

$$E_k := +\sqrt{\vec{k}^2 + m^2}$$

Green's Function in k-space

$$G(x, x') = \int d^4k \frac{-e^{-ik(x-x')}}{(\omega - E_k)(\omega + E_k)}$$

Split the d^4k into $d^3\vec{k}\; d\omega$

$$= \int d^3\vec{k} \left(-e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\right) \int_{-\infty}^{\infty} d\omega \underbrace{\frac{e^{-i\omega(t-t')}}{(\omega - E_k)(\omega + E_k)}}_{=: I_\omega}$$

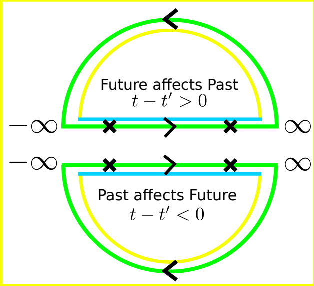
Using Contour Integration to evaluate

$$\int_{-\infty}^{\infty} d\omega \underbrace{\frac{e^{-i\omega(t-t')}}{(\omega - E_k)(\omega + E_k)}}_{=: I_\omega}$$

$$\oint_C I_\omega d\omega = \int_{-\infty}^{\infty} I_\omega d\omega + \int_{semi} I_\omega d\omega$$

Use Residue Theorem — Our Goal — = 0 by Jordan's Lemma

If $t - t' < 0$, close contour in upper half plane
If $t - t' > 0$, close contour in lower half plane



I_ω has 2 poles at $\omega = \pm E_k$

⊗ × $\quad R_- := \text{Res}(I_\omega, \omega = -E_k) = -\frac{e^{iE_k(t-t')}}{2E_k}$

× ⊗ $\quad R_+ := \text{Res}(I_\omega, \omega = +E_k) = \frac{e^{-iE_k(t-t')}}{2E_k}$

Contribution of Residues to Propagator

$$\int d^3\vec{k}\left(-e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\right) R_- = \int \frac{d^3\vec{k}}{2E_k} e^{-ik^\mu \cdot (x'_\mu - x_\mu)} \underbrace{=} \langle 0|\hat{\phi}(x')\hat{\phi}(x)|0\rangle$$

Canonical Quantisation

$$\int d^3\vec{k}\left(-e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\right) R_+ = -\int \frac{d^3\vec{k}}{2E_k} e^{-ik^\mu \cdot (x_\mu - x'_\mu)} \overbrace{=} -\langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle$$

Hadamard Regularization: 4 Choices of Contours	upper-half plane G_U $t - t' < 0$	lower-half plane G_L $t - t' > 0$	$G(x, x') = \int d^3\vec{k}\left(-e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\right)\left(\Theta(t'-t)G_U + \Theta(t-t')G_L\right)$
 Retarded Propagator	0	$-R_- - R_+$	$\Theta(t-t')\langle 0\|[\hat{\phi}(x), \hat{\phi}(x')]\|0\rangle = \Theta(t-t')\left(\langle 0\|\hat{\phi}(x)\hat{\phi}(x')\|0\rangle - \langle 0\|\hat{\phi}(x')\hat{\phi}(x)\|0\rangle\right)$
 Advanced Propagator	$R_- + R_+$	minus sign because contour goes clockwise 0	$\Theta(t'-t)\langle 0\|[\hat{\phi}(x'), \hat{\phi}(x)]\|0\rangle = \Theta(t'-t)\left(\langle 0\|\hat{\phi}(x')\hat{\phi}(x)\|0\rangle - \langle 0\|\hat{\phi}(x)\hat{\phi}(x')\|0\rangle\right)$
 Feynman Propagator	R_-	$-R_+$	$\langle 0\|\mathbf{T}\hat{\phi}(x)\hat{\phi}(x')\|0\rangle = \Theta(t'-t)\langle 0\|\hat{\phi}(x')\hat{\phi}(x)\|0\rangle + \Theta(t-t')\langle 0\|\hat{\phi}(x)\hat{\phi}(x')\|0\rangle$
	R_+	$-R_-$	$-\left(\langle 0\|\mathbf{T}\hat{\phi}(x)\hat{\phi}(x')\|0\rangle\right)^\dagger = -\Theta(t'-t)\langle 0\|\hat{\phi}(x)\hat{\phi}(x')\|0\rangle - \Theta(t-t')\langle 0\|\hat{\phi}(x')\hat{\phi}(x)\|0\rangle$